Soft Solids: Errata

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Chapter 1

Errata

Listed in this document are all errors known to the author in the Birkhäuser publication of his book *Soft Solids: A Primer to the Theoretical Mechanics of Materials* published in 2014.

To the reader: If you come across a typo or a more serious error in *Soft Solids*, please forward it to me. Your efforts will be greatly appreciated. You may contact me via email at `afreed@tamu.edu`.

Chapter 1

1) Students have suggested that homework problem 1.4.3, *Extension Followed by Simple Shear*, would be more intuitive if the assigned coordinate frame corresponded with simple shear, as put forward in 1.3.3, instead of uniaxial extension, as established in 1.3.1. To accommodate this request, the motion published in the book as

$$\mathsf{x}_1 = \lambda\, \mathsf{X}_1, \quad \mathsf{x}_2 = \gamma\lambda\, \mathsf{X}_1 + \lambda^{-n}\, \mathsf{X}_2, \quad \mathsf{x}_3 = \lambda^{n-1}\, \mathsf{X}_3 \tag{1.36}$$

whose inverse motion is given by

$$\mathsf{X}_1 = \lambda^{-1}\, \mathsf{x}_1, \quad \mathsf{X}_2 = -\gamma\lambda^{n}\, \mathsf{x}_1 + \lambda^{n}\, \mathsf{x}_2, \quad \mathsf{X}_3 = \lambda^{1-n}\, \mathsf{x}_3 \tag{1.37}$$

should be replaced by

$$\mathsf{x}_1 = \lambda^{-n}\, \mathsf{X}_1 + \gamma\lambda\, \mathsf{X}_2, \quad \mathsf{x}_2 = \lambda\, \mathsf{X}_2, \quad \mathsf{x}_3 = \lambda^{n-1}\, \mathsf{X}_3 \tag{1.36}$$

whose inverse motion is given by

$$\mathsf{X}_1 = \lambda^{n}\, \mathsf{x}_1 - \gamma\lambda^{n}\, \mathsf{x}_2, \quad \mathsf{X}_2 = \lambda^{-1}\, \mathsf{x}_2, \quad \mathsf{X}_3 = \lambda^{1-n}\, \mathsf{x}_3. \tag{1.37}$$

with Fig. 1.11 in the book being replaced by the following figure, and with the text immediately following Eq. (1.38) being replaced by "where the specimen's height aligns with the 2-direction and its width aligns with the 1-direction…".

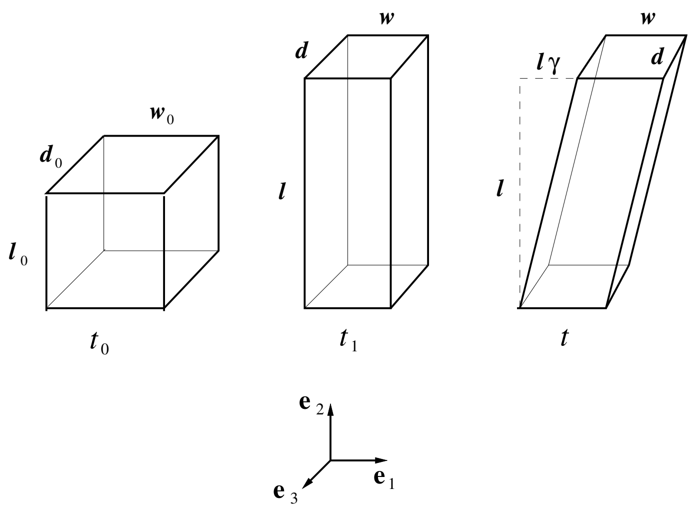

Fig. 1.11 Juxtaposition of an extension followed by a simple shear. Here ℓ_0, w_0, and d_0 are the dimensions of length, width, and depth of a gage section that is first extended to a rectangular prism with dimensions ℓ, w, and d, and later sheared by some extent γ.

Chapter 2

1) The paragraph of discussion following Eq. (2.4.2) is flawed. What appears in the book as

> By definition $\mathbb{r} \cdot \mathbb{r} = 1$ and, therefore, $2\,\mathbb{r} \cdot \dot{\mathbb{r}} = 0$ that, from Eq. (2.41), requires $\mathbb{r} \cdot \boldsymbol{A}^{-1}(\mathbb{\omega} - \dot{\theta}\,\mathbb{r}) = 0$. But it can be shown that $\mathbb{r} \cdot \boldsymbol{A}^{-1} = \csc(\theta)\,\mathbb{r} \neq \mathbb{0}$, which reflects the singularity of $\boldsymbol{A}$ present at $\theta = 0$, while $\mathbb{r} \cdot \boldsymbol{A} = \sin(\theta)\,\mathbb{r}$. Consequently, it is sufficient to require $(\mathbb{\omega} - \dot{\theta}\,\mathbb{r}) = \mathbb{0}$ to ensure $\mathbb{r} \cdot \boldsymbol{A}^{-1}(\mathbb{\omega} - \dot{\theta}\,\mathbb{r}) = 0$, thereby producing the anticipated result
>
> $$\mathbb{\omega} = \dot{\theta}\,\mathbb{r} \quad \therefore \quad \dot{\theta} = \mathbb{r} \cdot \mathbb{\omega} = \|\mathbb{\omega}\| \ \text{ with } \ \mathbb{r} = \frac{\mathbb{\omega}}{\|\mathbb{\omega}\|} \tag{2.43}$$
>
> which enables the rotation tensor $\mathbf{R}$ to be quantified via Eq. 2.39). To the best of the author's knowledge, $\mathbb{\omega} = \dot{\theta}\,\mathbb{r}$ has been suggested in the literature, but not rigorously proven.
>
> The axis of rotation $\mathbb{r}$ cannot be oriented in an absence of rotation, viz., whenever $\dot{\theta} = \|\mathbb{\omega}\| = 0. \ldots$

would be better explained if replaced by

> By definition $\mathbb{r} \cdot \mathbb{r} = 1$ and, therefore, $2\,\mathbb{r} \cdot \dot{\mathbb{r}} = 0$ implying that vector $\dot{\mathbb{r}}$ lies orthogonal to unit vector $\mathbb{r}$. It can be easily verified that $\mathbb{r} \cdot \boldsymbol{A} = \sin(\theta)\,\mathbb{r}$. Consequently, contracting the formulæ in Eq. (2.41) from the left by $\mathbb{r}$ leads to a pair of differential equations that one must solve, viz.,
>
> $$\dot{\theta} = \mathbb{r} \cdot \boldsymbol{\omega} \quad \text{and} \quad \dot{\mathbb{r}} = \boldsymbol{A}^{-1}\big(\boldsymbol{\omega} - (\mathbb{r} \cdot \boldsymbol{\omega})\mathbb{r}\big) \qquad (2.43)$$
>
> which describe the evolution of rotation $\mathbf{R}$ via its angle θ and axis $\mathbb{r}$ of rotation. Here matrix $\boldsymbol{A}^{-1}$ has components
>
> $$[\mathsf{A}^{-1}]_{11} = \big(1 + \mathsf{r}_1^2 + (1 - \mathsf{r}_1^2)\cos\theta\big)/2\sin\theta \qquad (2.44a)$$
> $$[\mathsf{A}^{-1}]_{12} = \big(\mathsf{r}_3\sin\theta + \mathsf{r}_1\mathsf{r}_2(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44b)$$
> $$[\mathsf{A}^{-1}]_{13} = -\big(\mathsf{r}_2\sin\theta + \mathsf{r}_1\mathsf{r}_3(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44c)$$
> $$[\mathsf{A}^{-1}]_{21} = -\big(\mathsf{r}_3\sin\theta + \mathsf{r}_1\mathsf{r}_2(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44d)$$
> $$[\mathsf{A}^{-1}]_{22} = \big(1 + \mathsf{r}_2^2 + (1 - \mathsf{r}_2^2)\cos\theta\big)/2\sin\theta \qquad (2.44e)$$
> $$[\mathsf{A}^{-1}]_{23} = \big(\mathsf{r}_1\sin\theta + \mathsf{r}_2\mathsf{r}_3(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44f)$$
> $$[\mathsf{A}^{-1}]_{31} = \big(\mathsf{r}_2\sin\theta + \mathsf{r}_1\mathsf{r}_3(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44g)$$
> $$[\mathsf{A}^{-1}]_{32} = -\big(\mathsf{r}_1\sin\theta + \mathsf{r}_2\mathsf{r}_3(1 - \cos\theta)\big)/2\sin\theta \qquad (2.44h)$$
> $$[\mathsf{A}^{-1}]_{33} = \big(1 + \mathsf{r}_3^2 + (1 - \mathsf{r}_3^2)\cos\theta\big)/2\sin\theta \qquad (2.44i)$$
>
> that possesses an obvious singularity at $\theta = 0$.
>
> The axis of rotation $\mathbb{r}$ cannot be oriented in an absence of rotation, viz., whenever $\theta = 0$ and $\boldsymbol{\omega} = \mathbb{0}$. . . .

As a consequence, Alg. 2.2 needs to be reworked—this is an outstanding action item.

Chapter 3

1) The discussion of Hencky strain in §3.2.1 is wrong, because the exponential of a matrix product equals the product of the matrix exponentials only when the matrices commute, and $\boldsymbol{RU}$ does not commute, in general. Therefore, replace the block of text found in the book

> One often hears of the terminology 'true strain,' a strain measure explored and developed by Hencky [1928, 1931] whose conceptual origin he attributes to Ludwig. It is a logical 1D strain measure, but it is not a practical 3D strain measure. Hencky strain is defined as $\boldsymbol{E}_H = \ln \boldsymbol{U}$ with an associated rotation tensor of $\mathbf{R}_H = \ln \mathbf{R}$ because $\ln \mathbf{F} = \ln(\mathbf{R}\boldsymbol{U}) = \ln \mathbf{R} + \ln \boldsymbol{U} = \mathbf{R}_H + \boldsymbol{E}_H$. These deformation fields are presented here only for the purpose of informing the reader about their existence. Taking the logarithm of a matrix is not easily done [Fitzgerald, 1980], nor is taking its rate [Hoger, 1986]. Its inverse operation, however, is described by a well-behaved convergent series: the exponential of a matrix. Specifically, one can write the identities[5]
>
> $$\begin{aligned} \boldsymbol{U} &= \exp(\boldsymbol{E}_H) &&= \boldsymbol{I} + \boldsymbol{E}_H + \tfrac{1}{2}\boldsymbol{E}_H^2 + \tfrac{1}{6}\boldsymbol{E}_H^3 + \cdots, \\ \boldsymbol{U}^{-1} &= \exp(-\boldsymbol{E}_H) &&= \boldsymbol{I} - \boldsymbol{E}_H + \tfrac{1}{2}\boldsymbol{E}_H^2 - \tfrac{1}{6}\boldsymbol{E}_H^3 + \cdots, \\ \mathbf{R} &= \exp(\mathbf{R}_H) &&= \mathbf{I} + \mathbf{R}_H + \tfrac{1}{2}\mathbf{R}_H^2 + \tfrac{1}{6}\mathbf{R}_H^3 + \cdots, \\ \mathbf{R}^{-1} &= \exp(-\mathbf{R}_H) &&= \mathbf{I} - \mathbf{R}_H + \tfrac{1}{2}\mathbf{R}_H^2 - \tfrac{1}{6}\mathbf{R}_H^3 + \cdots \end{aligned} \tag{3.18}$$
>
> from which one derives a useful pair of approximations
>
> $$\begin{aligned} \tfrac{1}{2}\big(\boldsymbol{U} - \boldsymbol{U}^{-1}\big) &= \boldsymbol{E}_H + \mathcal{O}\big(\tfrac{1}{3}\boldsymbol{E}_H^3\big), \\ \tfrac{1}{2}\big(\mathbf{R} - \mathbf{R}^{-1}\big) &= \mathbf{R}_H + \mathcal{O}\big(\tfrac{1}{3}\mathbf{R}_H^3\big) \end{aligned} \tag{3.19}$$
>
> that provide third-order accurate estimates for $\boldsymbol{E}_H$ and $\mathbf{R}_H$. Obviously, Hencky strain is a mixed tensor field [Freed, 1995], because $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ are both mixed tensor fields.

with the following block of text

> One often hears of the terminology 'true strain,' a strain measure that originated with Becker [1893], cf. Neff *et al.* [2014], which was later rediscovered and developed by Hencky [1928, 1931]. It is a logical 1D strain measure, but it is not a practical 3D strain measure [Freed, 2014]. Hencky strain is defined as $\boldsymbol{E}_H = \ln \boldsymbol{U}$. This deformation

[5] This clever formulation was shown to the author many years ago by Prof. Arkady Leonov. The author has not seen it published in the literature.

> field is presented here only for the purpose of informing the reader about its existence. Taking the logarithm of a matrix is not easily done [Fitzgerald, 1980; Freed & Srinivasa, 2015], nor is taking its rate easily calculated [Freed, 2014; Hoger, 1986]. Its inverse operation, however, is described by a well-behaved convergent series: the exponential of a matrix. Specifically, one can write the identities[5]
>
> $$\begin{aligned} \boldsymbol{U} &= \exp\left(\boldsymbol{E}_H\right) &&= \boldsymbol{I} + \boldsymbol{E}_H + \tfrac{1}{2}\boldsymbol{E}_H^2 + \tfrac{1}{6}\boldsymbol{E}_H^3 + \cdots , \\ \boldsymbol{U}^{-1} &= \exp\left(-\boldsymbol{E}_H\right) &&= \boldsymbol{I} - \boldsymbol{E}_H + \tfrac{1}{2}\boldsymbol{E}_H^2 - \tfrac{1}{6}\boldsymbol{E}_H^3 + \cdots , \end{aligned} \tag{3.18}$$
>
> from which one derives
>
> $$\tfrac{1}{2}\big(\boldsymbol{U} - \boldsymbol{U}^{-1}\big) = \boldsymbol{E}_H + \mathcal{O}\big(\tfrac{1}{3}\boldsymbol{E}_H^3\big), \tag{3.19}$$
>
> which is a useful, third-order, accurate estimate for $\boldsymbol{E}_H$. Obviously, Hencky strain is a mixed tensor field [Freed, 1995, 2014], because $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ are both mixed tensor fields.

which also attibutes the origin of logarithmic strain to Becker, whose lost work was rediscovered shortly after the book went to press.

Chapter 4

None known.

Chapter 5

None known.

Chapter 6

Equations (6.81 & 6.82) are in error by a factor of 2. They should read

$$\mathrm{ten}(\mathfrak{a}) = \frac{1}{E}\begin{bmatrix} 1 & 0 & 0 & -\nu \\ 0 & {}^{1+\nu}/_2 & {}^{1+\nu}/_2 & 0 \\ 0 & {}^{1+\nu}/_2 & {}^{1+\nu}/_2 & 0 \\ -\nu & 0 & 0 & 1 \end{bmatrix} - \frac{\beta}{2E}\begin{bmatrix} 2\mathsf{e}_{11} & \mathsf{e}_{12} & \mathsf{e}_{12} & 0 \\ \mathsf{e}_{12} & \frac{1}{2}(\mathsf{e}_{11} + \mathsf{e}_{22}) & \frac{1}{2}(\mathsf{e}_{11} + \mathsf{e}_{22}) & \mathsf{e}_{12} \\ \mathsf{e}_{12} & \frac{1}{2}(\mathsf{e}_{11} + \mathsf{e}_{22}) & \frac{1}{2}(\mathsf{e}_{11} + \mathsf{e}_{22}) & \mathsf{e}_{12} \\ 0 & \mathsf{e}_{12} & \mathsf{e}_{12} & 2\mathsf{e}_{22} \end{bmatrix} \tag{6.81}$$

[5] This clever formulation was shown to the author many years ago by Prof. Arkady Leonov. The author has not seen it published in the literature.

and

$$\mathrm{ten}(\boldsymbol{\mathfrak{b}}) = -\frac{\beta}{2E}\begin{bmatrix} 2\mathsf{s}_{11} & \mathsf{s}_{12} & \mathsf{s}_{12} & 0 \\ \mathsf{s}_{12} & \frac{1}{2}(\mathsf{s}_{11}+\mathsf{s}_{22}) & \frac{1}{2}(\mathsf{s}_{11}+\mathsf{s}_{22}) & \mathsf{s}_{12} \\ \mathsf{s}_{12} & \frac{1}{2}(\mathsf{s}_{11}+\mathsf{s}_{22}) & \frac{1}{2}(\mathsf{s}_{11}+\mathsf{s}_{22}) & \mathsf{s}_{12} \\ 0 & \mathsf{s}_{12} & \mathsf{s}_{12} & 2\mathsf{s}_{22} \end{bmatrix} \tag{6.82}$$

Chapter 7

None known.

Appendix A

None known.

Appendix B

None known.

Appendix C

None known.

Appendix D

1) There is a typo in Table D.3 in location U_{12}. What was reported as 0.25 should read 0.025. The corrected table is published below.

Appendix E

None known.

Appendix F

None known.

Table D.3 The partitioned matrix for an L-stable implicit IRKS method of third order that possesses the property F where $c = \{1/4, 1/2, 3/4, 1\}^T$ and whose coefficients for error estimation are $\phi_0 = 43.700369$ and $\phi = \{110.801474, -70.202212, -17.198526, 20.299631\}^T$.[a]

0.225	0	0	0	1	0.025	-0.05	-0.0265625
0.211287	0.225	0	0	1	0.063713	-0.0806435	-0.0833663
0.946338	-0.342943	0.225	0	1	-0.0783954	0.0947737	0.121956
0.52149	-0.662474	0.490476	0.225	1	0.425507	0.216014	-0.103603
0.52149	-0.662474	0.490476	0.225	1	0.425507	0.216014	-0.103603
0	0	0	1	0	0	0	0
-0.0423385	0.695379	-0.784079	1.0116	0	-0.880558	-0.521284	0.774748
0.077564	0.246379	-0.321806	0.274145	0	-0.276282	-0.350743	0.521284

[a]Parameters are for method *i3a* extracted from the *Atlas of general linear methods with inherent Runge-Kutta stability*, `http://www.math.auckland.ac.nz/~hpod/atlas`.

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